

A BRST gauge-fixing procedure for Yang–Mills theory on sphere

Rabin Banerjee*

*S.N. Bose National Centre for Basic Sciences, JD Block,
Sector III, Salt Lake City, Kolkata 700 098, India*

Shinichi Deguchi†

*Institute of Quantum Science,
College of Science and Technology,
Nihon University, Chiyoda-ku,
Tokyo 101-8308, Japan*

Abstract

A gauge-fixing procedure for the Yang–Mills theory on an n -dimensional sphere (or a hypersphere) is discussed in a systematic manner. We claim that Adler’s gauge-fixing condition used in massless Euclidean QED on a hypersphere is not conventional because of the presence of an extra free index, and hence is unfavorable for the gauge-fixing procedure based on the BRST invariance principle (or simply BRST gauge-fixing procedure). Choosing a suitable gauge condition, which is proved to be equivalent to a generalization of Adler’s condition, we apply the BRST gauge-fixing procedure to the Yang–Mills theory on a hypersphere to obtain consistent results. Field equations for the Yang–Mills field and associated fields are derived in manifestly $O(n+1)$ covariant or invariant forms. In the large radius limit, these equations reproduce the corresponding field equations defined on the n -dimensional flat space.

PACS numbers: 11.15.-q, 11.10.Ef, 11.10.Kk

Keywords: Adler’s gauge-fixing condition, BRST symmetry, conformal Killing vectors

* Email address: rabin@bose.res.in

† Email address: deguchi@phys.cst.nihon-u.ac.jp

I. INTRODUCTION

Manifestly $O(n + 1)$ -covariant formulation of gauge theories on an n -dimensional sphere (or a hypersphere) has been discussed in several contexts [1, 2, 3, 4, 5, 6, 7, 8]. Such a formulation was first developed by Adler to study massless Euclidean QED (quantum electrodynamics) on a hypersphere in 5-dimensional Euclidean space [1]. Infrared-finiteness of this theory was pointed out there as one of the advantages due to compactification of space-time. Further study of massless Euclidean QED on a hypersphere was made in a manifestly $O(n+1)$ -covariant way by using the dimensional regularization [2] and it was extended to the case of Yang–Mills theory [3]. The manifestly $O(5)$ -covariant formulation was also applied to analyzing classical and semi-classical behaviors of the pseudoparticle solution in the $SU(2)$ Yang–Mills theory [4, 5, 6]. The connection between the axial anomaly and the Atiyah–Singer index theorem was illustrated with the manifestly $O(n + 1)$ -covariant formulation in the cases of $n = 2$ and $n = 4$ [7]. Recently, the manifestly $O(n+1)$ -covariant formulation was reconsidered with the aid of conformal Killing vectors [8]. There, in addition to the Yang–Mills theory, a rank-2 antisymmetric tensor gauge theory and the case involving spinor fields were discussed by using formulas derived with the aid of conformal Killing vectors and spinors.

To study the quantum-theoretical properties of a gauge theory, we have to introduce a suitable gauge-fixing condition to this theory. In the study of massless Euclidean QED on a hypersphere, Adler adopted a gauge-fixing condition appropriate for manifestly $O(n + 1)$ -covariant analysis [1]. This condition makes the analysis quite simple and is applicable to higher-dimensions and to the case of Yang–Mills theory. However, Adler’s condition is unusual in the sense that it has an extra free index. For this reason, Adler’s condition is not favorable for the gauge-fixing procedure based on the Becchi–Rouet–Stora–Tyutin (BRST) invariance principle (or simply BRST gauge-fixing procedure) proposed by Kugo and Uehara [11, 12]. In fact, Adler’s condition has not been treated in connection with the first-order formalism of gauge-fixing [9] and with the BRST symmetry [10, 12].

The purpose of this paper is to apply the BRST gauge-fixing procedure to the Yang–Mills theory on a hypersphere in such a way that manifestly $O(n + 1)$ -covariance of the theory is maintained. To this end, we propose a gauge-fixing condition adapted for the BRST gauge-fixing procedure. Our condition is equivalent to a generalization of Adler’s condition, but has no extra free indices. A desirable gauge-fixing term is thus defined at the action level, and it is shown, with the aid of conformal Killing vectors, that this gauge-fixing term yields results compatible with those for the Yang–Mills theory on the flat space.

The paper is organized as follows: Section 2 is devoted to a brief review of manifestly $O(n + 1)$ -covariant formulation of the Yang–Mills theory on a hypersphere. Section 3 dis-

cusses the gauge-fixing conditions. Here the equivalence of a generalization of Adler's condition and our gauge-fixing condition without extra free indices is proved. Section 4 treats the BRST gauge-fixing procedure for the Yang–Mills theory on a hypersphere. In section 5, field equations for the Yang–Mills field and associated fields are derived in manifestly $O(n + 1)$ covariant or invariant forms. They are shown to reproduce, in the large radius limit, the corresponding field equations defined on the n -dimensional flat space. Section 6 contains concluding remarks.

II. YANG–MILLS THEORY ON HYPERSPHERE

In this section, we briefly review a manifestly $O(n + 1)$ -covariant formulation of the Yang–Mills theory on a hypersphere. Following the literature [1, 4, 5, 7, 8], we consider an n -dimensional unit sphere (or a unit hypersphere) S_1^n embedded in $(n + 1)$ -dimensional Euclidean space \mathbf{R}^{n+1} . The hypersphere S_1^n is characterized by the constraint $r_a r_a = 1$ imposed on Cartesian coordinates (r_a) ($a = 1, 2, \dots, n + 1$) on \mathbf{R}^{n+1} . One may use (r_μ) ($\mu = 1, 2, \dots, n$; $0 \leq r_\mu r_\mu \leq 1$) as local coordinates on S_1^n , treating $r_{n+1} = \pm\sqrt{1 - r_\mu r_\mu}$ as a dependent variable. In this treatment, the angular momentum operators L_{ab} read

$$L_{\mu\nu} = -i(r_\mu \partial_\nu - r_\nu \partial_\mu), \quad \partial_\mu \equiv \frac{\partial}{\partial r_\mu}, \quad (1)$$

$$L_{\mu(n+1)} = -L_{(n+1)\mu} = ir_{n+1}\partial_\mu, \quad (2)$$

or more concisely

$$L_{ab} = -i(r_a \partial_b - r_b \partial_a), \quad \partial_a \equiv \delta_{a\mu}\partial_\mu. \quad (3)$$

Noting that

$$\frac{\partial r_{n+1}}{\partial r_\mu} = -\frac{r_\mu}{r_{n+1}}, \quad (4)$$

one can show that the operators in Eqs. (1) and (2) satisfy the $O(n + 1)$ Lie algebra

$$[L_{ab}, L_{cd}] = i(\delta_{ac}L_{bd} - \delta_{bc}L_{ad} - \delta_{ad}L_{bc} + \delta_{bd}L_{ac}). \quad (5)$$

In terms of the stereographic coordinates (x_μ) which are related to (r_a) by

$$r_\mu = \frac{2x_\mu}{1 + x^2}, \quad r_{n+1} = \frac{1 - x^2}{1 + x^2}, \quad x \equiv \sqrt{x_\mu x_\mu}, \quad (6)$$

the angular momentum operators in Eqs. (1) and (2) are written as [8]

$$L_{ab} = -i(r_a K_b^\mu - r_b K_a^\mu) \frac{\partial}{\partial x_\mu}, \quad (7)$$

where K_a^μ are the conformal Killing vectors

$$K_\nu^\mu = \frac{1+x^2}{2} \delta_{\mu\nu} - x_\mu x_\nu, \quad K_{n+1}^\mu = -x_\mu. \quad (8)$$

Equation (6) illustrates a mapping from the flat space (or hyperplane) $\bar{\mathbf{R}}^n \equiv \mathbf{R}^n \cup \{\infty\}$ to S_1^n . The inverse mapping from S_1^n to $\bar{\mathbf{R}}^n$ is then illustrated by $x_\mu = r_\mu/(1+r_{n+1})$. The Killing vectors K_a^μ satisfy the transversality condition

$$r_a K_a^\mu = 0. \quad (9)$$

Let \hat{A}_a be a Yang–Mills field on S_1^n which takes values in a Lie algebra \mathfrak{g} ; \hat{A}_a can be expanded as $\hat{A}_a = \hat{A}_a^i T^i$ in terms of the Hermitian basis $\{T^i\}$ of \mathfrak{g} which satisfy the algebra $[T^i, T^j] = i f^{ijk} T^k$ with the structure constants f^{ijk} . The normalization conditions $\text{Tr}(T_i T_j) = \delta_{ij}$ are also put for convenience. One may regard \hat{A}_a as a function of the independent variables (r_μ) . The Yang–Mills field \hat{A}_a is assumed to live on the tangent space at a point $P(r_\mu)$ on S_1^n by imposing the transversality condition

$$r_a \hat{A}_a = 0. \quad (10)$$

This implies that one component of (\hat{A}_a) , for instance \hat{A}_{n+1} , depends on the other components, such as $\hat{A}_{n+1} = -(r_\mu \hat{A}_\mu)/r_{n+1}$. The differentiation of the condition (10) with respect to r_μ is carried out to get

$$r_b \partial_\mu \hat{A}_b = -\hat{A}_\mu + \frac{r_\mu}{r_{n+1}} \hat{A}_{n+1}. \quad (11)$$

Here Eq. (4) has been used. Equation (11) may be written as

$$r_b \partial_a \hat{A}_b = -\hat{A}_a + \frac{r_a}{r_{n+1}} \hat{A}_{n+1}, \quad (12)$$

because, in the case $a = \mu$, it reduces to Eq. (11) and, in the case $a = n + 1$, both sides of Eq. (12) vanish so that it gives rise to no additional conditions. (Note that ∂_a is understood to be $\delta_{a\mu} \partial_\mu$.)

The infinitesimal gauge transformation of \hat{A}_a is given by [8]

$$\delta_\lambda \hat{A}_a = ir_b \mathcal{L}_{ba} \lambda = P_{ab} \hat{D}_b \lambda, \quad (13)$$

where λ is an infinitesimal gauge parameter taking values in \mathfrak{g} , \mathcal{L}_{ab} are covariantized angular momentum operators

$$\begin{aligned} \mathcal{L}_{ab} &\equiv L_{ab} - [r_a \hat{A}_b - r_b \hat{A}_a,] \\ &= -i(r_a \hat{D}_b - r_b \hat{D}_a), \end{aligned} \quad (14)$$

while P_{ab} and D_a are the tangential projection operator and the covariant derivative, respectively:

$$P_{ab} \equiv \delta_{ab} - r_a r_b, \quad (15)$$

$$\hat{D}_a \equiv \partial_a - i[\hat{A}_a, \quad]. \quad (16)$$

The projection operator P_{ab} in Eq. (13) guarantees that the Yang–Mills field transformed according to the rule (13), i.e., $\hat{A}_a + \delta_\lambda \hat{A}_a$, lives on the same tangent space.

The field strength of \hat{A}_a can be written in a manifestly $O(n+1)$ -covariant form [4, 5, 8]:

$$\begin{aligned} \hat{F}_{abc} &= i(L_{ab}\hat{A}_c - r_a[\hat{A}_b, \hat{A}_c]) \\ &\quad + \text{cyclic permutations in } (a, b, c) \\ &= r_a \hat{F}_{bc} + r_b \hat{F}_{ca} + r_c \hat{F}_{ab}, \end{aligned} \quad (17)$$

where \hat{F}_{ab} is defined by

$$\hat{F}_{ab} = \partial_a \hat{A}_b - \partial_b \hat{A}_a - i[\hat{A}_a, \hat{A}_b]. \quad (18)$$

The gauge transformation of \hat{F}_{ab} is found from Eq. (13) to be

$$\begin{aligned} \delta_\lambda \hat{F}_{ab} &= -i[\hat{F}_{ab}, \lambda] + r_a \left(\hat{D}_b + \frac{1}{r_{n+1}} \delta_{b(n+1)} \right) (r_\mu \partial_\mu \lambda) \\ &\quad - r_b \left(\hat{D}_a + \frac{1}{r_{n+1}} \delta_{a(n+1)} \right) (r_\mu \partial_\mu \lambda), \end{aligned} \quad (19)$$

where Eqs. (4), (10) and (11) have been used. This is an inhomogeneous transformation involving terms that are not manifestly $O(n+1)$ covariant. Since field strengths should transform homogeneously, one cannot take \hat{F}_{ab} as the field strength of \hat{A}_a , even though \hat{F}_{ab} looks like the field strength as far as one sees only Eq. (18). Instead of \hat{F}_{ab} , the rank-3 tensor \hat{F}_{abc} transforms homogeneously under the gauge transformation,

$$\delta_\lambda \hat{F}_{abc} = -i[\hat{F}_{abc}, \lambda]. \quad (20)$$

Hence \hat{F}_{abc} has the property of field strength. With the field strength \hat{F}_{abc} , the Yang–Mills action for \hat{A}_a is given by

$$S_{\text{YM}} = \int d\Omega \left[-\frac{1}{12} \text{Tr}(\hat{F}_{abc} \hat{F}_{abc}) \right]. \quad (21)$$

Here $d\Omega$ is an invariant measure on S_1^n which is written in terms of the coordinates (r_μ) as

$$d\Omega = \frac{1}{|r_{n+1}|} \prod_{\mu=1}^n dr_\mu. \quad (22)$$

Obviously, the action S_{YM} is gauge invariant. The variation of S_{YM} can be calculated through integration by parts over (r_μ) , which is performed by taking into account the factor $|r_{n+1}|^{-1}$ contained in $d\Omega$. Using Eq. (4) and noting that S_1^n has no boundaries, we obtain

$$\delta S_{\text{YM}} = \int d\Omega \text{Tr} \left[\frac{i}{2} \delta \hat{A}_c \mathcal{L}_{ab} \hat{F}_{abc} \right]. \quad (23)$$

III. GAUGE-FIXING CONDITIONS

For studying quantum-theoretical structure of the Yang–Mills theory on the hypersphere S_1^n , it is necessary to consider the gauge-fixing procedure in the theory. We here focus our attention on discussing gauge-fixing conditions, before setting a suitable gauge-fixing term.

In a study of massless Euclidean QED on S_1^4 , Adler proposed a gauge-fixing condition $iL_{ab}\hat{A}_b = \hat{A}_a$ [1]. This is, of course, useful for making the analysis quite simple, and is also applicable to higher-dimensions as well as to the case of Yang–Mills theory. However, Adler’s condition is unusual in the sense that it has an extra free index a in comparison with the well-known Lorentz condition $\partial_\mu A_\mu = 0$; for this reason, Adler’s condition is not favorable to the ordinary first-order formalism of gauge-fixing [9, 12]. In this section, we shall prove that Adler’s condition is equivalent to the condition $ir_a L_{ab} \hat{A}_b = 0$ with no free indices. This condition, which has a form more similar to $\partial_\mu A_\mu = 0$ than Adler’s condition, is essentially the same as the one used in Refs. [5, 6] in a somewhat different context where the one-instanton background is present. Although Adler’s condition (or its quadratic equivalent) was compared with another condition $ir_a L_{ab} \hat{A}_b = 0$ in the literature [3], it seems that a complete proof of the equivalence has not been given yet.

To incorporate a gauge parameter α into the gauge-fixing conditions in a manifestly $O(n + 1)$ -covariant manner, we introduce the Nakanishi–Lautrup (NL) field \hat{B} on S_1^n [9]. Then the above-mentioned conditions are generalized as in Eqs. (24) and (25) given below. The generalization of Adler’s condition, Eq. (24), is also unusual in the sense that it has a free index a . We now show the following.

Proposition: The following conditions (a) and (b) are equivalent.

$$(a) \quad iL_{ab}\hat{A}_b + \alpha r_a \hat{B} = \hat{A}_a, \quad [*] \quad (24)$$

$$(b) \quad ir_a L_{ab} \hat{A}_b + \alpha \hat{B} = 0. \quad (25)$$

Proof: Consider the condition (a). Contracting Eq. (24) by r_a yields Eq. (25), owing to the constraint $r_a r_a = 1$ and the condition (10). The condition (b) is thus derived from (a).

[*] Using Eqs. (10) and (12), we can readily show that $\hat{A}_a(iL_{ab}\hat{A}_b - \hat{A}_a) = 0$, which is compatible with the

Next we shall derive (a) from (b). Using Eq. (12), $r_a r_a = 1$ and the condition (10), one can rewrite Eq. (25) as

$$\partial_b \hat{A}_b = \frac{1}{r_{n+1}} \hat{A}_{n+1} - \alpha \hat{B}. \quad (26)$$

This is an alternative form of the condition (b). Substituting Eqs. (12) and (26) into

$$iL_{ab} \hat{A}_b = r_a \partial_b \hat{A}_b - r_b \partial_a \hat{A}_b \quad (27)$$

leads us to Eq. (24). Consequently, the equivalence of the conditions (a) and (b) is established. \square

Since Eqs. (24) and (25) hold simultaneously, we can eliminate the NL field \hat{B} from Eq. (24) with the aid of Eq. (25), getting

$$iL_{ab} \hat{A}_b - \hat{A}_a = r_a (ir_c L_{cb} \hat{A}_b). \quad (28)$$

From this formula, we readily see that the condition $ir_a L_{ab} \hat{A}_b = 0$ is equivalent to Adler's condition $iL_{ab} \hat{A}_b = \hat{A}_a$. Adding $\alpha r_a \hat{B}$ to the both sides of Eq. (28), we have

$$iL_{ab} \hat{A}_b - \hat{A}_a + \alpha r_a \hat{B} = r_a (ir_c L_{cb} \hat{A}_b + \alpha \hat{B}). \quad (29)$$

This formula converts the conditions (a) and (b) into each other. Equation (29) will also be useful for simplifying field equations.

IV. BRST SYMMETRY AND A GAUGE-FIXING TERM

It is known that gauge-fixing is neatly performed by considering the BRST invariance as a first principle [11, 12]. In the previous section, we have claimed that Adler's condition and its generalization, namely the condition (a), are not conventional owing to the presence of a free index. If one applies the BRST gauge-fixing procedure to the Yang–Mills theory on S_1^n , the condition (b) is much better to use than (a), though (a) and (b) are equivalent. The reason for choosing (b) is that it resembles the ordinary gauge-fixing condition $\partial_\mu A_\mu + \alpha B = 0$ which has often been adopted in the BRST gauge-fixing procedure [11, 12]. With the condition

condition (a). The equation derived here implies that the vector $(iL_{ab} \hat{A}_b - \hat{A}_a)$ is null or perpendicular to the vector (A_a) living in a tangent space of S_1^n . Whereas $(iL_{ab} \hat{A}_b - \hat{A}_a)$ can live in the same tangent space, the condition (a) requires that $(iL_{ab} \hat{A}_b - \hat{A}_a)$ is null or in the normal (or radial) direction, being perpendicular to S_1^n .

(b), it is easy to apply the conventional procedure to the present case since there is no free index.

The BRST transformation δ is defined for \hat{A}_a by replacing λ in Eq. (13) by the Faddeev–Popov (FP) ghost field \hat{C} on S_1^n ,

$$\delta\hat{A}_a = ir_b \mathcal{L}_{ba} \hat{C} = P_{ab} \hat{D}_b \hat{C}. \quad (30)$$

By putting the transformation rule

$$\delta\hat{C} = \frac{i}{2} \{\hat{C}, \hat{C}\}, \quad (31)$$

the nilpotency property $\delta^2 = 0$ is guaranteed for \hat{A}_a and \hat{C} . In particular, $\delta^2 \hat{A}_a = 0$ is verified by using the property $P_{ac}P_{cb} = P_{ab}$. In addition to \hat{C} , the FP anti-ghost field $\hat{\bar{C}}$ is introduced to satisfy

$$\delta\hat{\bar{C}} = i\hat{B}, \quad \delta\hat{B} = 0. \quad (32)$$

Consequently the nilpotency property $\delta^2 = 0$ is still held after incorporating \hat{C} and \hat{B} . Needless to say, \hat{C} , $\hat{\bar{C}}$ and \hat{B} are treated as functions of (r_μ) .

With the BRST transformation and the relevant fields in hand, we can discuss the BRST gauge-fixing procedure for the Yang–Mills theory on S_1^n . To deal with the condition (b) in a BRST symmetric manner, we now take the sum of gauge-fixing and FP ghost terms written in the following form: [†]

$$S_{\text{GF}} = \int d\Omega \left\{ -i\delta \text{Tr} \left[\hat{\bar{C}} \left(ir_a L_{ab} \hat{A}_b + \frac{\alpha}{2} \hat{B} \right) \right] \right\}. \quad (33)$$

Since this is a BRST-coboundary term, the BRST invariance of S_{GF} is guaranteed due to the nilpotency of δ . In contrast, the BRST invariance of S_{YM} is clear from its gauge invariance.

[†] It is possible to deal with Adler’s gauge-fixing condition by taking

$$\tilde{S}_{\text{GF}} = \int d\Omega \left\{ -i\delta \text{Tr} \left[\hat{\bar{C}}_a \left(iL_{ab} \hat{A}_b - \hat{A}_a + \frac{\alpha}{2} \hat{B}_a \right) \right] \right\}$$

instead of Eq. (33). Here \hat{B}_a and $\hat{\bar{C}}_a$ are a NL field and a FP anti-ghost field of the vector type, respectively, satisfying $\delta\hat{\bar{C}}_a = i\hat{B}_a$, $\delta\hat{B}_a = 0$. The necessity of introducing the vector type of NL and FP anti-ghost fields is due to the fact that Adler’s gauge-fixing condition has a free index. Although \tilde{S}_{GF} works well as a sum of gauge-fixing and FP ghost terms, it involves redundant degrees of freedom caused by the vectorial property of \hat{B}_a and $\hat{\bar{C}}_a$. As a result, the discussion based on \tilde{S}_{GF} is fairly complicated. Also, in this case, the symmetry between a FP ghost field and a FP anti-ghost field is spoiled. This turns out to be a serious problem when one applies the superfield formalism [13] of the BRST and anti-BRST symmetries to the Yang–Mills theory on S_1^n [14].

Carrying out the BRST transformation contained in the right-hand side of Eq. (33) and using the formula

$$r_a L_{ac} (r_b \mathcal{L}_{bc}) = \frac{1}{2} L_{ab} \mathcal{L}_{ab}, \quad (34)$$

we have

$$S_{\text{GF}} = \int d\Omega \text{Tr} \left[\hat{B} i r_a L_{ab} \hat{A}_b + \frac{\alpha}{2} \hat{B}^2 - \frac{1}{2} i \hat{C} L_{ab} \mathcal{L}_{ab} \hat{C} \right]. \quad (35)$$

Integrating by parts over (r_μ) and using Eqs. (4), (10) and $r_a r_a = 1$, we may rewrite Eq. (35) as

$$\begin{aligned} S_{\text{GF}} &= \int d\Omega \text{Tr} \left[- (ir_b L_{ba} \hat{B}) \hat{A}_a + \frac{\alpha}{2} \hat{B}^2 + (ir_b L_{ba} \hat{C}) r_c \mathcal{L}_{ca} \hat{C} \right] \\ &= \int d\Omega \text{Tr} \left[- (ir_b L_{ba} \hat{B} - \{ir_b L_{ba} \hat{C}, \hat{C}\}) \hat{A}_a + \frac{\alpha}{2} \hat{B}^2 + (ir_b L_{ba} \hat{C}) r_c L_{ca} \hat{C} \right] \end{aligned} \quad (36)$$

Further integration by parts for the ghost term in Eq. (36) leads to

$$S_{\text{GF}} = \int d\Omega \text{Tr} \left[- (ir_b L_{ba} \hat{B}) \hat{A}_a + \frac{\alpha}{2} \hat{B}^2 - \frac{1}{2} i (\mathcal{L}_{ab} L_{ab} \hat{C}) \hat{C} \right], \quad (37)$$

with use of the formula

$$r_a \mathcal{L}_{ac} (r_b L_{bc}) = \frac{1}{2} \mathcal{L}_{ab} L_{ab}. \quad (38)$$

Quantization of the fields \hat{A}_a , \hat{C} , $\hat{\bar{C}}$ and \hat{B} is performed based on the total action

$$S = S_{\text{YM}} + S_{\text{GF}} \quad (39)$$

in a systematic way; thereby one may see quantum-theoretical structure of the Yang–Mills theory on S_1^n . Details of the quantization will be discussed elsewhere.

V. FIELD EQUATIONS

From the total action S , we can derive the Euler–Lagrange equation for each field,

$$\frac{i}{2} \mathcal{L}_{ab} \hat{F}_{abc} = ir_b L_{bc} \hat{B} - \{ir_b L_{bc} \hat{C}, \hat{C}\}, \quad (40)$$

$$ir_a L_{ab} \hat{A}_b + \alpha \hat{B} = 0, \quad (41)$$

$$L_{ab} \mathcal{L}_{ab} \hat{C} = 0, \quad (42)$$

$$\mathcal{L}_{ab} L_{ab} \hat{C} = 0. \quad (43)$$

The field equation (40), which is directly found from Eqs. (23) and (36), is just the Yang–Mills equation on S_1^n including the additional terms due to S_{GF} . By contracting Eq. (40) by r_c , both sides of this equation vanish, implying that Eq. (40) satisfies the transversality condition. If the right-hand side of Eq. (40) is put to be zero, it reduces to the Yang–Mills equation obtained by Jackiw and Rebbi [4]. Equation (41), which is easily read-off from Eq. (35), is precisely the condition (b). The field equations (42) and (43) follow from Eqs. (35) and (37), respectively. From Eq. (40), the field equation governing the motion of \hat{B} is obtained as

$$\mathcal{L}_{ab} L_{ab} \hat{B} = \{L_{ab} \hat{\bar{C}}, \mathcal{L}_{ab} \hat{C}\} \quad (44)$$

after using Eq. (43) and $r_a r_a = 1$. In the Abelian case, using the commutation relations (5) and $[L_{ab}, r_c] = -i(r_a \delta_{bc} - r_b \delta_{ac})$, as well as Eqs. (29) and (41), we can rewrite Eq. (40) as

$$L_{ab} L_{ab} \hat{A}_c + 2(n-2) \hat{A}_c + 4\alpha r_c \hat{B} + 2i(1-\alpha) r_b L_{bc} \hat{B} = 0. \quad (45)$$

Even in the Feynman gauge $\alpha = 1$, the NL field \hat{B} still remains in the third term of Eq. (45). This is a difference between the present case and the ordinary case formulated on the flat space. Contracting Eq. (45) by r_c reproduces Eq. (41) by virtue of the presence of the third term, which fact demonstrates compatibility of the field equation (45) and the gauge condition (41). In the particular situation of $\hat{B} = 0$, Eq. (45) in four dimensions reduces to the field equation found by Adler [1].

The field equations Eqs. (40)–(44) have manifestly $O(n+1)$ covariant or invariant forms. They are the spherical analogues of the field equations on the flat space given in the literature [10, 12]:

$$D_\mu F_{\mu\nu} = \partial_\nu B - \{\partial_\nu \bar{C}, C\}, \quad (46)$$

$$\partial_\mu A_\mu + \alpha B = 0, \quad (47)$$

$$\partial_\mu D_\mu C = 0, \quad (48)$$

$$D_\mu \partial_\mu \bar{C} = 0, \quad (49)$$

$$D_\mu \partial_\mu B = \{\partial_\mu \bar{C}, D_\mu C\}, \quad (50)$$

where

$$D_\mu \equiv \partial_\mu - i[A_\mu, \quad], \quad (51)$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \quad (52)$$

Comparing Eqs. (40)–(44) with Eqs. (46)–(50), we see that L_{ab} and \mathcal{L}_{ab} correspond to ∂_μ and D_μ , respectively. In the Yang–Mills theory on S_1^n , L_{ab} and \mathcal{L}_{ab} are more fundamental than ∂_μ and D_μ . This can be understood from the fact that translations on a plane are realized as rotations on a sphere, so that usual derivatives are replaced by angular derivations when one discusses on a sphere. It is possible to establish the correspondence between Eqs. (40)–(44) and Eqs. (46)–(50) through the following discussion.

The Yang–Mills field \hat{A}_a on the hypersphere S_1^n and the conventional one A_μ on the flat space $\bar{\mathbf{R}}^n$ is related by the conformal Killing vectors K_a^μ [8]:

$$\hat{A}_a(r) = K_a^\mu A_\mu(x). \quad (53)$$

The field strength \hat{F}_{abc} can be expressed as [8]

$$\hat{F}_{abc} = (r_a K_b^\mu K_c^\nu + r_b K_c^\mu K_a^\nu + r_c K_a^\mu K_b^\nu) F_{\mu\nu}, \quad (54)$$

with $F_{\mu\nu}$ defined by Eq. (52). Similarly to Eq. (53), the fields \hat{B} , \hat{C} and $\hat{\bar{C}}$ on S_1^n are related to the corresponding fields B , C and \bar{C} on $\bar{\mathbf{R}}^n$ by

$$\hat{B}(r) = \left(\frac{1+x^2}{2}\right)^2 B(x), \quad \hat{C}(r) = \frac{1+x^2}{2} C(x), \quad \hat{\bar{C}}(r) = \frac{1+x^2}{2} \bar{C}(x). \quad (55)$$

These relations may be understood from a supersphere formulation of the Yang–Mills theory on S_1^n [14]. From Eq. (7), it follows that

$$ir_b L_{ba} = K_a^\mu \partial_\mu, \quad \partial_\mu \equiv \frac{\partial}{\partial x_\mu}, \quad (56)$$

where $r_a r_a = 1$ and the condition (9) has been used. A covariantized version of Eq. (56) is also satisfied owing to Eq. (10),

$$ir_b \mathcal{L}_{ba} = K_a^\mu D_\mu, \quad (57)$$

with D_μ defined by Eq. (51). We now rewrite the field equations (40)–(44) in terms of the relevant fields on $\bar{\mathbf{R}}^n$. To this end, the following formulas are particularly useful:

$$K_a^\mu K_a^\nu = \left(\frac{1+x^2}{2}\right)^2 \delta_{\mu\nu}, \quad (58)$$

$$K_a^\mu \partial_\mu K_a^\nu = (2-n) \left(\frac{1+x^2}{2}\right) x_\nu, \quad (59)$$

$$\begin{aligned} & K_a^\rho \partial_\rho (K_a^\mu K_b^\nu - K_a^\nu K_b^\mu) \\ &= (4-n) \frac{1+x^2}{2} (x_\mu K_b^\nu - x_\nu K_b^\mu). \end{aligned} \quad (60)$$

Substituting Eqs. (53)–(55) into Eqs. (40)–(44) and using the formulas (58)–(60), together with $r_a r_a = 1$ and Eqs. (9), (10), (34), (38), (56) and (57), at appropriate stages, we can write Eqs. (40)–(44) in terms of the stereographic coordinates:

$$D_\mu F_{\mu\nu} + \frac{2(4-n)}{1+x^2} x_\mu F_{\mu\nu} = \partial_\nu B + \frac{4x_\nu}{1+x^2} B - \left\{ \partial_\nu \bar{C} + \frac{2x_\nu}{1+x^2} \bar{C}, C \right\}, \quad (61)$$

$$\partial_\mu A_\mu + \frac{2(2-n)}{1+x^2} x_\mu A_\mu + \alpha B = 0, \quad (62)$$

$$\partial_\mu D_\mu C + \frac{2x_\mu}{1+x^2} \{(3-n)D_\mu C + \partial_\mu C\} + \frac{2(4-n)x^2 + 2n}{(1+x^2)^2} C = 0, \quad (63)$$

$$D_\mu \partial_\mu \bar{C} + \frac{2x_\mu}{1+x^2} \{(3-n)\partial_\mu \bar{C} + D_\mu \bar{C}\} + \frac{2(4-n)x^2 + 2n}{(1+x^2)^2} \bar{C} = 0, \quad (64)$$

$$D_\mu \partial_\mu B + \frac{2x_\mu}{1+x^2} \{(4-n)\partial_\mu B + 2D_\mu B\} + \frac{4(6-n)x^2 + 4n}{(1+x^2)^2} B = \left\{ \partial_\mu \bar{C} + \frac{2x_\mu}{1+x^2} \bar{C}, D_\mu C + \frac{2x_\mu}{1+x^2} C \right\}. \quad (65)$$

Equation (62) may be regarded as a combination of the ordinary gauge-fixing condition $\partial_\mu A_\mu + \alpha B = 0$ and the Fock–Schwinger gauge condition $x_\mu A_\mu = 0$ [15], involving an extra factor $(1+x^2)^{-1}$. In the case of $\alpha = 0$, Eq. (62) is essentially a higher-dimensional analogue of the gauge-fixing condition proposed by Ore [5]. An analogue of Eq. (63) is also seen in Ref. [5]. In the present paper, however, Eqs. (62) and (63) have been derived systematically by considering the BRST symmetry and using the conformal Killing vectors.

We now make a replacement of r_a by r_a/R to explicitly incorporate the radius R of a hypersphere into the field equations. Thereby, the hypersphere S_1^n is scaled to be S_R^n characterized by $r_a r_a = R^2$, and the Yang–Mills theory in question is formulated on S_R^n . By taking the limit $R \rightarrow \infty$, the hypersphere S_R^n approximates to the flat space, i.e. $S_\infty^n = \mathbf{R}^n$. To maintain the relation between two coordinate systems (r_μ) and (x_μ) even in the large radius limit, x_μ has to be replaced by x_μ/R . Correspondingly, the derivative gets replaced as $\partial_\mu \mapsto R\partial_\mu$. Accordingly, the relevant fields on the flat space have to be modified appropriately. For instance, to maintain homogeneity of the changes, $A_\mu \mapsto RA_\mu$, leading to $F_{\mu\nu} \mapsto R^2 F_{\mu\nu}$. Any other transformation for A_μ would lead to different transformations for the derivative piece and the commutator piece in $F_{\mu\nu}$ leading to a lack of homogeneity. Similarly, the other fields are modified as, $B \mapsto R^2 B$, $C \mapsto RC$ and $\bar{C} \mapsto R\bar{C}$. After making the replacement in Eqs. (61)–(65), one can take the limit $R \rightarrow \infty$ to see that Eqs. (61)–(65) reduce to Eqs. (46)–(50), respectively; the irrelevant terms in Eqs. (61)–(65) vanish on taking the limit. Therefore Eqs. (46)–(50) are recognized to be the large radius limit of Eqs. (61)–(65).

We can also express Eq. (45) as an Abelian field equation for A_μ and B written in terms of the stereographic coordinates. After making the replacement mentioned above in this equation, the term corresponding to the third term of Eq. (45) vanishes in the limit $R \rightarrow \infty$. Hence, in the Feynman gauge $\alpha = 1$, the NL field B is completely removed and the Abelian field equation turns out to be the Laplace equation $\partial_\mu \partial_\mu A_\nu = 0$.

VI. CONCLUSIONS

After briefly reviewing a manifestly $O(n + 1)$ -covariant formulation of the Yang–Mills theory on the hypersphere S_1^n , we have discussed a gauge-fixing procedure for this theory by considering the BRST invariance as a first principle. It was stressed that although Adler’s gauge-fixing condition is useful for concrete analyses, it is unfavorable for the BRST gauge-fixing procedure owing to the presence of an extra free index. Instead of Adler’s condition, we proposed a suitable gauge-fixing condition which is equivalent to a generalization of Adler’s condition, but has no extra free indices. A complete proof of the equivalence was given by introducing the NL field.

Having obtained the suitable gauge-fixing condition, the BRST gauge-fixing procedure was applied to the Yang–Mills theory on S_1^n , toward investigating its quantum-theoretical properties. The gauge-fixing and FP ghost terms, as well as the Yang–Mills action, were written in manifestly $O(n + 1)$ -invariant forms with the aid of the angular momentum operators and their gauge-covariantized versions. Consequently, the field equation for each of the Yang–Mills, NL, FP ghost, and FP anti-ghost fields was derived in a manifestly $O(n + 1)$ covariant or invariant form. All the field equations were also written in terms of the stereographic coordinates using the conformal Killing vectors which act like a metric in translating formulas from the local coordinates (r_μ) to the stereographic coordinates (x_μ) and vice-versa. Then it was shown that these equations reduce to corresponding field equations defined on the n -dimensional flat space, in the limit where the radius of the hypersphere is taken very large.

In quantizing the relevant fields in the Yang–Mills theory on the hypersphere S_R^n , one can expect that infrared divergences are automatically regularized, because S_R^n is a bounded space with a maximum length. For this reason, the Yang–Mills theory on S_R^n would be appropriate for studying properties of quantum chromodynamics (QCD) at a low-energy region, such as the gluon condensate of dimension 2 [16]. Also, the BRST symmetry discussed in this paper may be suitable for the geometric approach to non-Abelian chiral anomalies with the use of decent equations and BRST cohomology [17], because, there, space-time is assumed to be essentially the hypersphere S_R^n . In such an approach, non-Abelian chiral

anomalies will be found in association with the axial anomaly (or singlet anomaly) evaluated on a hypersphere [7, 8].

Acknowledgments

One of the authors (S.D.) would like to thank Prof. K. Fujikawa for his encouragements. Also, R.B. thanks the Nihon University for providing support and the members of the Institute of Quantum Science for their gracious hospitality during his visit when this work was started.

- [1] S. L. Adler, Phys. Rev. D 6 (1972) 3445; Phys. Rev. D 8 (1973) 2400; hep-ph/0505177.
- [2] I. T. Drummond and G. M. Shore, Ann. Phys. 117 (1979) 89.
- [3] G. M. Shore, Ann. Phys. 117 (1979) 121.
- [4] R. Jackiw and C. Rebbi, Phys. Rev. D 14 (1976) 517.
- [5] F. R. Ore, Jr, Phys. Rev. D 15 (1977) 470.
- [6] A. A. Belavin and A. M. Polyakov, Nucl. Phys. B 123 (1977) 429.
- [7] N. K. Nielsen and B. Schroer, Nucl. Phys. B 127 (1977) 493.
- [8] R. Banerjee, Ann. Phys. 311 (2004) 245.
- [9] N. Nakanishi, Prog. Theor. Phys. 35 (1966) 1111;
B. Lautrup, Kgl. Danske Videnskab. Selskabs. Mat.-fys. Medd. 35, No. 11 (1967).
- [10] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl No.66 (1979) 1.
- [11] T. Kugo and S. Uehara, Nucl. Phys. B 197 (1982) 378.
- [12] N. Nakanishi and I. Ojima, Covariant Operator Formalism of Gauge Theories and Quantum Gravity, World Scientific, Singapore, 1990.
- [13] L. Bonora and M. Tonin, Phys. Lett B 98 (1981) 48;
R. Delbourgo and P. D. Jarvis, J. Phys. A 15 (1982) 611.
- [14] R. Banerjee and S. Deguchi, in preparation.
- [15] V. A. Fock, Sov. Phys. 12 (1937) 404;
J. Schwinger, Phys. Rev. 82 (1951) 664;
See also, J. Barcelos-Neto, C. A. P. Galvão and P. Gaete, Mod. Phys. Lett. A 6 (1991) 1597.

- [16] F. V. Gubarev, L. Stodolsky, V. I. Zakharov, Phys. Rev. Lett. 86 (2001) 2220;
F. V. Gubarev and V. I. Zakharov, Phys. Lett. B 501 (2001) 28.
- [17] R. Stora, in *New developments in quantum field theory and statistical mechanics*. M. Lévy and P. Mitter (eds.), Plenum Press (1976) p. 201;
B. Zumino, in *Relativity, groups and topology II*. B. S. DeWitt and R. Stora (eds.), North-Holland (1984) p. 1291;
B. Zumino, Y.-S. Wu and A. Zee, Nucl. Phys. B 239 (1984) 477.